priors on exchangeable directed graphs

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Overview

• We consider **exchangeable (dense) graphs**.

• Specifically, we are interested in **directed** graphs. However, most work has focused on undirected graphs.

• **Aldous–Hoover** has an analogous statement for directed graphs, which is more complicated than merely using an *asymmetric* function.

• Many natural nonparametric priors on exchangeable undirected graphs **extend to the directed case**.

• This perspective leads to natural priors on other exchangeable structures, such as **tournaments** and **directed acyclic graphs**.
Exchangeable graphs

exchangeability:

- order of vertices doesn’t affect distribution of graph

\[ \Pr(\begin{array}{ccc}
1 & 2 & 3 \\
\end{array}) = \Pr(\begin{array}{ccc}
3 & 2 & 1 \\
\end{array}) \]
## Exchangeable graphs

<table>
<thead>
<tr>
<th>Undirected</th>
<th>Directed</th>
</tr>
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<td>Any exchangeable random infinite graph is obtained as a mixture of $G(\infty, W)$.</td>
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Exchangeable graphs

Graphon:

\[ W : [0, 1]^2 \rightarrow [0, 1] \]
\[ W(x, y) = W(y, x) \]

\[ W(x, y) = \frac{(1 - x) + (1 - y)}{2} \]
Exchangeable graphs

Graphon:

\[ W : [0, 1]^2 \to [0, 1] \]

\[ W(x, y) = W(y, x) \]

Sampling procedure:

\[ U_i \sim \text{Uniform}[0, 1] \]

\[ G_{ij} \sim \text{Bern}(W(U_i, U_j)), i < j \quad \text{(Set } G_{ji} = G_{ij}) \]
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Exchangeable digraphs

Many results for exchangeable undirected graphs extend to directed graphs.
Exchangeable digraphs

Many results for exchangeable **undirected** graphs extend to **directed** graphs.

- order of vertices doesn’t affect distribution of graph

\[
\Pr(\begin{array}{ccc}
\text{1} & \text{2} & \text{3} \\
\end{array}) = \Pr(\begin{array}{ccc}
\text{1} & \text{2} & \text{3} \\
\end{array})
\]

[Cai–Ackerman–Freer, 2015]
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<td>$W = ?$</td>
</tr>
<tr>
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<td>implicit in A–H; cf. Diaconis–Janson</td>
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Exchangeable digraphs

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Exchangeable digraphs

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Yes, by independently choosing each edge direction.
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But this does not cover all directed graphs:
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Yes, by independently choosing each edge direction.

But this does not cover \textit{all} directed graphs:

- \textbf{tournaments} (each pair of vertices has exactly one directed edge)
Exchangeable digraphs

Can we just use a single asymmetric measurable function?

Yes, by independently choosing each edge direction.

But this does not cover all directed graphs:

• **tournaments** (each pair of vertices has exactly one directed edge)

• **undirected** graphs (each pair has either both or no directed edges)
Digraph representation

\[ W_{00} \quad W_{01} \quad W_{10} \quad W_{11} \quad w \]

- Digraphon: \( W = (W_{00}, W_{01}, W_{10}, W_{11}, w) \)
  
  \( W_{ab} : [0, 1]^2 \to [0, 1] \) and \( w : [0, 1] \to \{0, 1\} \) measurable functions

- For \( a, b \in \{0, 1\} \) and \( x, y \in [0, 1] \),
  
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  3. \( \sum_{a,b} W_{a,b}(x, y) = 1 \)
Digraph representation

\[
\begin{array}{cccc}
W_{00} & W_{01} & W_{10} & W_{11} \\
0 & 1 & 1 & 1 \\
1 & & & \\
\end{array}
\]

\[w\]
Digraph representation

\[ W_{00} \quad W_{01} \quad W_{10} \quad W_{11} \quad w \]

\[
\begin{array}{ccc}
0 & 1 \\
1 & \quad & \quad
\end{array}
\]

\[
\begin{array}{ccc}
\quad & \quad & \quad \\
\quad & \quad & \quad
\end{array}
\]

\[
\begin{array}{ccc}
\quad & \quad & \quad \\
\quad & \quad & \quad
\end{array}
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0 & & \\
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\[ w \]

Digraph representation

\[ W_{00}, W_{01}, W_{10}, W_{11} \]

\[ w \]

\[
\begin{array}{c}
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0 \\
1
\end{array}
\end{array}
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• Define $W_4(x, y) := (W_{00}(x, y), W_{01}(x, y), W_{10}(x, y), W_{11}(x, y))$

$U_i \sim \text{Uniform}[0, 1]$

$E_{ij} \sim \text{Categorical}(W_4(U_i, U_j)), \text{ for } i < j$

$G_{ij}, G_{ji} = \text{directed edge(s) corresponding to } E_{ij}, \text{ for } i < j$

$G_{ii} = w(U_i)$
Sampling procedure

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priors on exchangeable directed graphs

[Cai–Ackerman–Freer, 2015]
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Special cases

- Undirected graphs (graphon) (e.g., ER(½))

\[
W_{00} \quad W_{01} \quad W_{10} \quad W_{11}
\]

\[
W(x, y) = W(y, x)
\]

\[
W : [0, 1]^2 \to [0, 1]
\]
Special cases

• **Tournaments** (e.g., generic tournament)

\[
\begin{align*}
W_{00} & \quad W_{01} & \quad W_{10} & \quad W_{11} & \quad w \\
\end{align*}
\]

- W_{00} (empty graph)
- W_{01} (complete graph)
- W_{10} (complete graph)
- W_{11} (empty graph)
- w (any directed graph)
Special cases

• Linear ordering  (the only one, by Glasner–Weiss)

\[ W_{00} \quad W_{01} \quad W_{10} \quad W_{11} \quad w \]
Special cases

- Directed acyclic graphs (DAGs)

\[
\begin{align*}
W_{00} & \quad W_{01} & \quad W_{10} & \quad W_{11} & \quad w \\
\end{align*}
\]
Special cases

- Partial ordering (poset)

\[
W_{00} \quad W_{01} \quad W_{10} \quad W_{11} \quad w
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Priors on digraphons

- Can extend literature for graphon priors (cf. Orbanz–Roy) to directed graphs, e.g., infinite relational model

- Some of these models are already intended for directed graphs, via an *asymmetric* measurable function (to describe independent edge directions).

[Orbanz–Roy, 2015]
Directed block models

• In a block model, pairs of regions can vary in how tournament-like, as well as how dense they are:

• e.g., Directed stochastic block model [Wang–Wong, 1987]

**Example of SBM digraphon**, 0.7 division

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Infinite digraph block model

1. Draw partition $\sim \text{DP-Stick}(\alpha)$
2. Get weights $\sim \text{Dirichlet}(a)$

$W_{00} \quad W_{01} \quad W_{10} \quad W_{11}$
Infinite digraph block model

1. Draw partition $\sim$ DP-Stick($\alpha$)
Infinite digraph block model

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Infinite digraph block model

Tournament

Dirichlet parameter $a = [0, 2, 1, 0]$

sample
Infinite digraph block model

Almost totally ordered

Dirichlet parameter $a = [0, 0, 1, 0]$

sample

reordered

[Cai–Ackerman–Freer, 2015]
Infinite digraph block model

Dirichlet parameter $a = [0.9, 2.0, 1.0, 0.5]$
Infinite digraph block model

- Inference via collapsed Gibbs sampling of cluster assignments
  \[ p(z_i | z_{-i}, G) \propto p(z_i | z_{-i}) p(G | z_i, z_{-i}) \]
  - cluster assignment of vertex i
  - cluster assignments of all vertices except i

- Experiments using synthetic data
  Dirichlet parameter \( a = [1, 1, 1, 1] \)
Infinite digraph block model

- Random digraphon

- Random sample, 100 vertices

Dirichlet parameter $a = [1, 1, 1, 1]$
Infinite digraph block model

• Random digraphon

• Collapsed Gibbs sampling

Dirichlet parameter \( a = [1, 1, 1, 1] \)
Infinite digraph block model

• Random digraphon

\[ \text{Dirichlet parameter } a = [1, 1, 1, 1] \]

• Collapsed Gibbs sampling

original ordering

resorted by increasing uniform random variables

resorted by inferred cluster

[Cai–Ackerman–Freer, 2015]
Infinite digraph block model

- Random digraphon: ER + tournament
Infinite digraph block model

- Schematic of the sampled digraph:

```
  1  2
1  1  1
2  2  2
```

- Diagram with groups 1 and 2:

```
  group 1  1/2  1/2  group 2
```

```
  1/2  or  1/2
```

[Cai–Ackerman–Freer, 2015]
Infinite digraph block model

- Random digraphon: ER + tournament

- Collapsed Gibbs sampling for this model

original ordering
resorted by increasing uniform random variables
resorted by inferred cluster
Infinite digraph block model

• Random digraphon: ER + tournament

• Collapsed Gibbs sampling for this model

original ordering

resorted by increasing uniform random variables

resorted by inferred cluster
Infinite digraph block model

• Random digraphon: ER + tournament

• Collapsed Gibbs sampling for the infinite relational model

original ordering
resorted by increasing uniform random variables
resorted by inferred cluster
Conclusions

• Summary

• Discussion:
  • other types of block models:
    • could consider other partitions, e.g., Pitman–Yor
  • other priors, e.g., Gaussian process (as in Lloyd et al.)
  • Aldous–Hoover already describes exchangeable hypergraphs
  • sparsity
References


